Modeling Loan Static Pools

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A static pool of loans is a grouping of loans with similar credit risk characteristics that were originated during a specific time period. These pools are tracked by the lending institutions with regard to defaults, credit losses, contractual payments, and non-contractual prepayments over the term. In this white paper we will model a static pool of loans. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

ABC Bank originates loans on a monthly basis over multiple product types (mortgages, home equity loans, auto loans, consumer loans, etc.). We are given the following go-forward model assumptions with respect to one pool of loans of a given product type...

Table 1: Go-Forward Model Assumptions

Description	Value
Loan originations in month 15 ($)	1,000,000
Loan pool weighted-average life in years (#)	4.00
Ratio of capital to assets (%)	10.00
Return on capital (%)	12.00

Our task is to answer the following questions...

Question 1: What is loan pool principal value at the end of month 36?

Question 2: What is after-tax net income of this loan pool in year 4?

Question 3: What is net cash flow of this loan pool in year 4?

Loan Pool Principal Balance

We will define the following variables to be time in years...

$$0 \leq s \leq t \leq \infty \text{ ...and... } 0 \leq s \leq m \leq n \leq \infty \tag{1}$$

Using the time variables in Equation (1) above we will define the variable $P(s)_t$ to be the principal balance at time t of a loan pool originated at time s, and the variable λ to be the loan amortization rate, which is a function of the loan pool's weighted average life. The equation for loan pool principal balance is...

$$P(s)_t = P(s)_s \operatorname{Exp}\left\{ -\lambda\,(t-s)\right\} \tag{2}$$

The derivative of Equation (2) above with respect to time is...

$$\frac{\delta}{\delta t} P(s)_t = -\lambda\, P(s)_s \operatorname{Exp}\left\{ -\lambda\,(t-s)\right\} \text{ ...such that... } \delta P(s)_t = -\lambda\, P(s)_s \operatorname{Exp}\left\{ -\lambda\,(t-s)\right\} \delta t \tag{3}$$

We will define the variable $P(s)_{m,n}$ to be cumulative principal reductions over the time interval $[m,n]$ on a loan pool originated at time s. Using Equation (3) above the equation for cumulative principal reductions is...

$$P(s)_{m,n} = \int\limits_m^n \delta P(s)_u = P(s)_n - P(s)_m \tag{4}$$

Note that cumulative principal reductions in Equation (4) above is a negative number because loan pool principal balance decreases over time such that the first derivative of loan pool principal balance with respect to time (Equation (3) above) is a negative number.

We will define the variable ω to be the loan pool's weighted average life in years. Using Equation (3) above the equation for the model parameter λ is... [1]

$$\text{if... } \omega = \int\limits_s^\infty (u - s)\, \delta P(s)_u \div \int\limits_s^\infty \delta P(s)_u \text{ ...then... } \lambda = \frac{1}{\omega} \tag{5}$$

Capital Investment

We will define the variable $X(s)_t$ to be capital investment at time t applicable to a loan pool originated at time s, and the variable ϕ to be the capital ratio, which is assumed to be a constant. To maximize the return on capital, which is a metric that determines bank value, the capital ratio is kept as high as possible but within regulatory and lender constraints. Using Equation (2) above the equation for capital investment is...

$$X(s)_t = \phi\, P(s)_t = \phi\, P(s)_s \,\mathrm{Exp}\left\{ -\lambda\,(t-s) \right\} \tag{6}$$

The derivative of Equation (6) above with respect to time is...

$$\frac{\delta}{\delta t} X(s)_t = -\lambda\,\phi\, P(s)_s \,\mathrm{Exp}\left\{ -\lambda\,(t-s) \right\} \text{ ...such that... } \delta X(s)_t = -\lambda\,\phi\, P(s)_s \,\mathrm{Exp}\left\{ -\lambda\,(t-s) \right\} \delta t \tag{7}$$

We will define the variable $X(s)_{m,n}$ to be cumulative capital investment over the time interval $[m, n]$ on a loan pool originated at time s. Using Equations (6) and (7) above the equation for cumulative capital investment is...

$$X(s)_{m,n} = \int\limits_m^n \delta X(s)_u = \phi \left(P(s)_n - P(s)_m \right) \tag{8}$$

Note that cumulative capital investment in Equation (8) above is a negative number. Given that loan pool principal balance is decreasing over time the capital investment applicable to that loan pool is also decreasing over time, which means that capital is being returned to the bank to be deployed elsewhere.

Net Income

We will define the variable $I(s)_t$ to be after-tax net income realized over the time interval $[t, t + \delta t]$ on a loan pool originated at time s and the variable π to be the after-tax return on capital. Using Equation (6) above the equation for after-tax net income is...

$$I(s)_t = \pi\,\phi\, P(s)_t\, \delta t = \pi\,\phi\, P(s)_s \,\mathrm{Exp}\left\{ -\lambda\,(t-s) \right\} \delta t \tag{9}$$

We will define the variable $I(s)_{m,n}$ to be cumulative after-tax net income realized over the time interval $[m, n]$ on a loan pool originated at time s. Using Equation (9) above the equation for cumulative after-tax net income is...

$$I(s)_{m,n} = \int\limits_m^n I(s)_u\, \delta u = \pi\,\phi\, P(s)_s \,\mathrm{Exp}\left\{ \lambda\, s \right\} \int\limits_m^n \mathrm{Exp}\left\{ -\lambda\, u \right\} \delta u \tag{10}$$

Using Appendix Equation (20) below the solution to Equation (10) above is...

$$I(s)_{m,n} = \pi\,\phi\, P(s)_s \,\mathrm{Exp}\left\{ \lambda\, s \right\} \left(\mathrm{Exp}\left\{ -\lambda\, m \right\} - \mathrm{Exp}\left\{ -\lambda\, n \right\} \right) \lambda^{-1} \tag{11}$$

Net Cash Flow

We will define the variable $C(s)_t$ to be net cash flow realized over the time interval $[t, t+\delta t]$ on a loan pool originated at time s. Using Equations (6) and (9) above the equation for net cash flow is...

$$C(s)_t = I(s)_t + X(s)_t = \phi\,(\pi + \lambda)\,P(s)_s\,\mathrm{Exp}\left\{-\lambda\,(t-s)\right\}\delta t \tag{12}$$

We will define the variable $C(s)_{m,n}$ to be cumulative net cash flow realized over the time interval $[m, n]$ on a loan pool originated at time s. Using Equation (12) above the equation for cumulative net cash flow is...

$$C(s)_{m,n} = \int\limits_m^n C(s)_u\,\delta u = \phi\,(\pi+\lambda)\,P(s)_s \int\limits_m^n \mathrm{Exp}\left\{-\lambda\,(u-m)\right\}\delta u = \phi\,(\pi+\lambda)\,P(s)_s\,\mathrm{Exp}\left\{\lambda\,m\right\}\int\limits_m^n \mathrm{Exp}\left\{-\lambda\,u\right\}\delta u \tag{13}$$

Using Appendix Equation (20) below the solution to Equation (13) above is...

$$C(s)_{m,n} = \phi\,(\pi+\lambda)\,P(s)_s\,\mathrm{Exp}\left\{\lambda\,m\right\}\left(\mathrm{Exp}\left\{-\lambda\,m\right\} - \mathrm{Exp}\left\{-\lambda\,n\right\}\right)\lambda^{-1} \tag{14}$$

The Answers To Our Hypothetical Problem

Using Equation the model assumptions in Table 1 above...

Table 2: Model Parameters

Symbol	Description	Value
$P(s)_s$	Loan originations in month 15	1,000,000
ω	Loan pool weighted-average life in years	4.0000
ϕ	Ratio of capital to assets	0.1000
π	Return on capital	0.1200

Using Equation (5) above the value of the variable λ is...

$$\lambda = \frac{1}{4.00} = 0.25 \tag{15}$$

We will define the following time variable values...

$$s = \frac{15}{12} = 1.25 \ \text{...and...}\ t = \frac{36}{12} = 3.00 \ \text{...and...}\ m = \frac{36}{12} = 3.00 \ \text{...and...}\ n = \frac{48}{12} = 4.00 \tag{16}$$

Question 1: What is loan pool principal value at the end of year 3?

Using Equation (2) above the answer to the question is...

$$P(1.25)_{3.00} = 1,000,000 \times \mathrm{Exp}\left\{-0.25 \times (3.00 - 1.25)\right\} = 645,649 \tag{17}$$

Question 2: What is after-tax net income of this loan pool in year 4?

Using Equation (11) above the answer to the question is...

$$\begin{aligned} I(1.25)_{3,4} &= 0.12 \times 0.10 \times 1,000,000 \times \mathrm{Exp}\left\{0.25 \times 1.25\right\} \times \left(\mathrm{Exp}\left\{-0.25 \times 3\right\} - \mathrm{Exp}\left\{-0.25 \times 4\right\}\right) \times 0.25^{-1} \\ &= 6,855 \end{aligned} \tag{18}$$

Question 3: What is net cash flow of this loan pool in year 4?

Using Equation (14) above the answer to the question is...

$$\begin{aligned} C(1.25)_{3,4} &= 0.10 \times (0.12 + 0.25) \times 1,000,000 \times \mathrm{Exp}\left\{0.25 \times 1.25\right\} \times \left(\mathrm{Exp}\left\{-0.25 \times 3\right\} - \mathrm{Exp}\left\{-0.25 \times 4\right\}\right) \times 0.25^{-1} \\ &= 21,137 \end{aligned} \tag{19}$$

Appendix

A. The solution to the following integral is...

$$\int\limits_s^t \mathrm{Exp}\left\{-\lambda\,u\right\}\delta u = -\frac{1}{\lambda}\,\mathrm{Exp}\left\{-\lambda\,u\right\}\Bigg[_s^t = \left(\mathrm{Exp}\left\{-\lambda\,s\right\} - \mathrm{Exp}\left\{-\lambda\,t\right\}\right)\lambda^{-1} \tag{20}$$

References

[1] Gary Schurman, *Integration By Parts - Weighted-Average Revenue Life,* January, 2020.